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SELF-SIMILAR SOLUTIONS OF THE BELLMAN EQUATION FOR OPTIMAL CORRECTION OF RANDOM DISTURBANCES

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A nonlinear second order partial differential equation (Bellman equation) is solved for some characteristic problems of optimal correction of motion in the presence of random distrubances and integral constraints on the control function.

For these problems, classes of self-similar (invariant group) solutions of the Bellman equation are computed. Some exact analytical solutions are obtained.

1. Formulation of problem. Let the motion of the system be described by the following equation:

$$dx / dt = a(t)u + b(t)\xi, \qquad x(t_0) = x_0 \tag{1.1}$$

Here t is time, x is the scalar phase coordinate, u is the control function, ξ is the random disturbance which is represented by white noise of constant intensity, $a(t)$ and $b(t)$ are given functions of time which have the meaning of control efficiency and intensity of disturbance, respectively, t_0 and x_0 are incidental conditions. The problem consists of determining the control which satisfies the constraint

$$\int_{t_0}^{T} |u|^m dt \leqslant q_0, \quad m > 0, \quad q_0 \geqslant 0 \tag{1.2}$$

and which minimizes the mathematical expectation of the following function of the phase coordinate at the end of the process:

$$J = F(x(T)) \tag{1.3}$$

Here T is the given time of process termination, q_0 is a given number, m is a fixed parameter.

The function $F(x)$ is a measure of deviation of the system from zero. Let us assume that it has even properties, that it is nonnegative and strictly monotonic, namely:

$$F(x) = F(-x), \quad F(0) = 0, \quad F(x) > 0 \ (x \neq 0), \quad F'(x) > 0 \ (x > 0) \tag{1.4}$$

where the prime indicates a derivative. The control u will be determined in the form of a system, i.e. as a function of arbitrary initial conditions t_0, x_0 and q_0, where $t_0 \leqslant T$, and $q_0 \geqslant 0$.

It is assumed that these variables can be measured exactly at any given instant. We note that the case $m = 1$ corresponds to a control of motion with a constraint on fuel consumption, while the case $m = 2$ corresponds to a control with a constraint on energy (in particular, control by means of low thrust).

Let us introduce the variable q which has the significance of available capability for control. From relationship (1.2) we obtain the equation and the boundary conditions for q in the form

$$dq / dt = -|u|^m, \qquad q(t_0) = q_0 \geqslant 0, \qquad q(T) \geqslant 0 \tag{1.5}$$

Without destroying generality we can assume that $a(t) \geqslant 0$ and $b(t) \geqslant 0$ for all t. If any of these functions assume a negative value for some t, then for this t we can change the sign of the function u or ξ in Eq. (1.1). All functions which are used are assumed to have the required number of derivatives.

We note that the described formulation of the problem is typical for a fairly wide class of problems of optimal control. In fact, let the motion be described by the general non-linear system

$$dX / dt = f(X, U, \xi, t) \tag{1.6}$$

where X is the vector of phase coordinates, U is the vector of control functions, ξ is the vector of disturbances and f is a given vector function. Let us denote by $Y = F_0(X, t)$ the vector of independent first integrals of system (1.6) for $U = \xi = 0$, then the following equality is valid:

$$\frac{\partial F_0(X, t)}{\partial X} f(X, 0, 0, t) + \frac{\partial F_0(X, t)}{\partial t} = 0 \tag{1.7}$$

where $\partial F_0 / \partial X$ is a square matrix of partial derivatives and $\partial F_0 / \partial t$ is a vector. Let the control U and the disturbances ξ in the system (1.6) be small in magnitude. Then we obtain from Eqs. (1.6), (1.7)

$$\frac{dY}{dt} = \frac{\partial F_0}{\partial X} [f(X, U, \xi, t) - f(X, 0, 0, t)] \approx \left(\frac{\partial F_0}{\partial X} \frac{\partial f}{\partial U}\right) U + \left(\frac{\partial F_0}{\partial X} \frac{\partial f}{\partial \xi}\right) \xi \tag{1.8}$$

Here the matrices of partial derivatives should be taken for $U = 0$, $\xi = 0$ and $X = X_0(t)$ where $X_0(t)$ is some reference trajectory which plays the role of undisturbed motion, i.e. it is any perticular solution of system (1.6) for $U = \xi = 0$. In this connection the matrix coefficients for U and ξ in system (1.8) are functions of t only, and this system is the vector analog of Eq. (1.1). If the criterion which is to be minimized depends only on one component of vector Y, then we can examine separately only one equation from the system (1.8), i.e. the equation (1.1).

For example, let us examine the similarity problem of correction of one-dimensional motion of a point. The equations of motion have the form

$$dX_1/dt = X_2, \qquad dX_2 / dt = u + \xi \tag{1.9}$$

where X_1 is the coordinate, X_2 is the velocity, u is the control force, ξ is the disturbance force. Let us introduce a new variable (T is the instant of process termination)

$$x = X_1 + (T - t) X_2 \tag{1.10}$$

It is easy to verify that this variable is the first integral of system (1.9) for $u = \xi = 0$. From relationships (1.9) and (1.10) we obtain

$$dx / dt = (T - t) u + (T - t) \xi \tag{1.11}$$

If the criterion which is to be minimized at the end of the process depends only on the coordinate $X_1(T)$, then by virtue of the relationship $X_1(T) = x(T)$ we can examine instead of system (1.9), one equation (1.11). Under corresponding conditions and constraints on the control we arrive at the problem (1.1)–(1.3) formulated above.

2. Bellman equation and boundary conditions. Let us designate by $S(t, x, q)$ the minimum value of the functional, i.e. of the mathematical expectation of function (1.3), which can be achieved for initial conditions $t_0 = t$, $x_0 = x$, $q_0 = q$ in the problem (1.1)–(1.3). It is apparent that always $S \geqslant 0$.

At first let us examine the case $0 < m < 1$. Let the control $u(t)$ differ from zero only in the interval $[t_1 - \varepsilon, t_1]$ and be equal to ε^{-1} in this interval. In this connection $\varepsilon > 0$ and $t_1 < T$. For $\varepsilon \to 0$ the control tends toward the delta function $\delta(t - t_1)$. It is evident from Eq. (1.1) that this leads to a finite jump of the phase coordinate $x(t)$ by the quantity $a(t_1)$ at the instant t_1. However, the integral (1.2) for this control tends to zero for $\varepsilon \to 0$, if $m < 1$.

In this manner a control of the type of a delta function does not lead an expenditure of the capability q for $m < 1$. It follows from this, that for any $q_0 > 0$, any t_0, x_0 and any realization of the random process $\xi(t)$ it is possible to achieve the equality $x(T)=0$ by means of a control impulse at the end of the process, i.e.

$$u(t) = - x(T - 0)\, a^{-1}(T)\, \delta(t - T) \tag{2.1}$$

where $x(T - 0)$ is the value of the phase coordinate directly before the application of the impulse. The control (2.1) ensures an absolute (zero) minimum of the functional to be minimized (mathematical expectation of function (1.3)). Therefore this control is optimal, but of course not unique. If $a(T) = 0$, the control (2.1) loses its meaning. However, if function $a(t)$ is different from zero for $t < T$, then it is possible to construct a minimizing sequence of impulse equations which also realizes a zero minimum of the functional. In this manner for the case $m < 1$ the problem (1.1)–(1.3) is solved trivially and here $S(t, x, q) \equiv 0$ for $t < T$.

From now on we shall assume $m \geqslant 1$. At first let us examine the case $m > 1$. Taking into account Eqs. (1.1), (1.5), we construct the Bellman equation for the function S [1]

$$S_t + \min_u [a(t) u S_x - |u|^m S_q] + \frac{1}{2} b^2(t) S_{xx} = 0 \tag{2.2}$$

Subscripts associated with S denote partial derivatives. We note the following properties of function S which result from the formulation of problem (1.1)–(1.3) and properties (1.4)

$$S(t, x, q) = S(t, -x, q), \quad \operatorname{sign} S_x = \operatorname{sign} x, \quad S_q < 0 \tag{2.3}$$

The first property expresses the evenness of function S. The second and third properties follow from the fact that the greater the initial deviation $|x_0|$ in (1.1) and the smaller the reserve of capability q_0 in (1.2), the greater will be the final deviation $|x(T)|$, for other conditions being equal. The minimum with respect to u in (2.2) is reached for

$$u = -[a(t) S_x / (-mS_q)]^{1/(m-1)} \operatorname{sign} x \tag{2.4}$$

After computation of the minimum taking into account (2.3), (2.4), Eq. (2.2) takes the following form in the region $x \geqslant 0$:

$$S_t + {}^1/_2 b^2(t) S_{xx} + (m-1)[a(t) S_x / (-mS_q)]^{m/(m-1)} S_q = 0 \tag{2.5}$$

On the boundary $q = 0$ the function S has a singularity. Let us fix $t < T$, $x > 0$ and let us take the limit $q_0 = q \to 0$, assuming that almost everywhere $a > 0$ and $S_x > 0$. Then the optimal control u of the problem (1.1)–(1.3) in the interval $[t, T]$ will tend to zero. Here with respect to order of magnitude $|u|^m \sim q$. Comparing this relationship with Eq. (2.4), we obtain

$$|S_q| = O(q^{(1-m)/m}), \qquad q \to 0 \tag{2.6}$$

It is convenient to make a substitution of variables (it is assume that $b > 0$)

$$\tau = \int_t^T b^2(t_1)\, dt_1, \qquad p^m = q \tag{2.7}$$

It follows from (2.6) and (2.7) that the derivative of S_p is bounded for $p \to 0$, and in the new variables now the function $S(\tau, x, p)$ does not have a singularity for $p \to 0$. Equation (2.5) in variables (2.7) takes the form

$$S_\tau = \frac{1}{2} S_{xx} + \frac{(m-1)p}{mb_1^2(\tau)} \left[\frac{a_1(\tau) S_x}{-S_p}\right]^{m/(m-1)} S_p, \quad m > 1 \tag{2.8}$$

The following initial and boundary conditions for the function $S(\tau, x, p)$ follow from the relationships (1.3), (2.3) and (2.7):

$$S(0, x, p) = F(x), \qquad S_x(\tau, 0, p) = 0 \tag{2.9}$$

For $p = 0$ Eq. (2.8) transforms into the usual equation of thermal conductivity. For conditions (2.9) the solution of this equation has the form

$$S(\tau, x, 0) = \frac{1}{\sqrt{2\pi\tau}} \int_{-\infty}^{\infty} \exp\left[-\frac{(x-x_1)^2}{2\tau}\right] F(x_1)\, dx_1 \tag{2.10}$$

Here the evenness of the function $F(x)$ is utilized. We note that solution (2.10) corresponds to uncontrolled motion. The capability of control p is equal to zero. In this manner the problem has been reduced to the determination of function $S(\tau, x, p)$ in the region $D = \{\tau \geqslant 0, x \geqslant 0, p \geqslant 0\}$ from Eq. (2.8) and boundary conditions (2.9), (2.10) on the boundaries of the region. Strictly speaking, the relationship (2.10) is not a boundary condition, because it follows from Eq. (2.8) itself. We can show that, generally speaking, there are enough given conditions in order to solve the problem in steps with respect to τ in the direction of increasing τ, for example by the finite difference method.

A difficulty is encountered for small τ, because for $\tau \to 0$ we have $S \to F(x)$ and therefore $S_p \to 0$. In this connection the second term in the right side (2.8) tends to ∞ for $\tau \to 0$. Let us find the asymptotic solution of Eq. (2.8) for small τ. The first term in the right side of (2.8) tends to $F''(x)/2$ for $\tau \to 0$. This term is bounded and therefore it can be neglected in comparison to the second term. Then Eq. (2.8) takes the form

$$S_\tau = \frac{(m-1)p}{mb_1^2(\tau)}\left[\frac{a_1(\tau)F'(x)}{-S_p}\right]^{m/(m-1)} S_p, \qquad \tau \to 0 \tag{2.11}$$

where the derivative S_x is replaced by its limiting value $F'(x)$. The solution of Eq. (2.11) which satisfies the initial condition (2.9) is sought in the form

$$S(\tau, x, p) = F(x) - pF'(x) f_1(\tau), \; f_1(0) = 0 \tag{2.12}$$

Substituting Eq. (2.12) into (2.11), we obtain the equation for the determination of function f_1

$$\frac{df_1}{d\tau} = \frac{m-1}{m} a_1^{m/(m-1)}(\tau) b_1^{-2}(\tau) f_1^{-1/(m-1)}(\tau)$$

Integrating this equation with the initial condition $f_1(0) = 0$, we obtain

$$f_1(\tau) = \left[\int_0^\tau \frac{a_1^{m/(m-1)}(\tau_1)}{b_1^2(\tau_1)} d\tau_1\right]^{(m-1)/m} \tag{2.13}$$

Relationships (2.12) and (2.13) determine the asymptotics of function S for $\tau \to 0$. and make it possible to "retreat" from the plane $\tau = 0$, providing a solution for small $\tau > 0$, which is important for numerical computation. Having determined the solution of the boundary value problem (2.8)–(2.10), it is then possible to find the system of optimal control. According to Eqs. (2.4), (2.7) we have

$$u(\tau, x, p) = -p[-a_1(\tau) S_x / S_p]^{1/(m-1)} \; (x > 0), \; u(\tau, -x, p) = -u(\tau, x, p) \tag{2.14}$$

3. The case $m = 1$. The important case $m = 1$ requires special examination because equations in Sect. 2 contain indeterminacies for $m \to 1$. We introduce the notation

$$Q = a_1(\tau) S_x + S_p \tag{3.1}$$

and rewrite Eq. (2.8) in the form

$$S_\tau = \frac{1}{2} S_{xx} + \frac{(m-1)p}{mb_1^2(\tau)}\left(1 - \frac{Q}{S_p}\right)^{m/(m-1)} S_p \tag{3.2}$$

Let us go to the limit for $m \to 1+0$, assuming that the function S for $m = 1$ is sufficiently smooth and taking into account that $S_p < 0$. If in some point $Q > 0$, then the limit of the right side of Eq. (3.2) for $m \to 1$ turns out to be equal to infinity, which has no meaning. Therefore we have $Q \leqslant 0$ for $m = 1$ everywhere in the region D. Let us denote by D_1 that part of region D where $Q < 0$, and by D_2 the remaining part, where $Q = 0$. In the region D_1, going to the limit for $m \to 1$ in Eq. (3.2), we obtain

$$S_\tau = {}^1/_2 S_{xx}, \quad Q = a_1(\tau) S_x + S_p < 0 \;\; \text{, in } D_1 \tag{3.3}$$

In the region D_2 we have $Q \equiv 0$, in which case it follows from (3.1) that

$$S(\tau, x, p) = G[x - a_1(\tau)p, \; \tau], \quad Q = 0 \;\; \text{in } D_2 \tag{3.4}$$

where G is an arbitrary function of two variables, subject to determination.

Let us denote by Γ the boundary of regions D_1 and D_2. If the function $S(\tau, x, p)$ has been found in region D_1, then on the boundary Γ, in accordance with continuity of function S, we shall have the initial condition $S = G$ on Γ. From this condition the function G and the complete solution (3.4) is determined in region D_2. For the solution of Eq. (3.3) in region D_1 it is necessary to give two boundary conditions on the unknown boundary Γ. One condition, $Q = 0$ on Γ, follows from the continuity of Q on Γ. In order to obtain the second condition, we differentiate both parts of Eq. (3.2) with respect to x and then set $Q = 0$. We obtain

$$\left(S_\tau - \frac{1}{2} S_{xx}\right)_x = \frac{p}{b_1^2(\tau)} \left(\frac{m-1}{m} S_{px} - \frac{Q_x}{S_p}\right) \tag{3.5}$$

Equation (3.5) is valid for $m > 1$ on the surface $Q = 0$, which for $m \to 1$ becomes the boundary Γ. But for $m \to 1$ the left part of relationship (3.5) tends to zero on Γ, because in region D_1 which is adjacent to Γ, Eq. (3.3) is identically satisfied. Therefore, going to the limit for $m \to 1$, we obtain from (3.5) that $Q_x = 0$ on Γ. Analogously $Q_p = 0$ on Γ. However, only one of the two conditions $Q_x = 0$ and $Q_p = 0$ is independent, because $Q \equiv 0$ on Γ.

In this manner, for $m = 1$ the wanted function S satisfies Eqs. (3.3) and (3.4) in regions D_1 and D_2. On the boundary Γ between these regions S is continuous and satisfies the conditions

$$Q = a_1(\tau) S_x + S_p = 0, \quad Q_x = a_1(\tau) S_{xx} + S_{px} = 0 \quad \text{on } \Gamma \tag{3.6}$$

Furthermore, conditions (2.9), (2.10) are valid on the boundaries of region D. From the second condition (2.9) and the inequality $S_p < 0$ it follows that $Q < 0$ for $x = 0$, i.e. the plane $x = 0$ belongs to region D_1. We note also that $p = q$ for $m = 1$ according to (2.7) and that $b_1(\tau)$ does not enter into the formulation of the problem (3.3), (3.4), (3.6).

Regions D_1 and D_2 have a simple meaning. From (2.14) for $m \to 1$ it is easy to obtain that $u = 0$ in the region D_1, i.e. in region D_1 uncontrolled motion takes place under the influence of only random distrubance. In the region D_2, on the other hand, impulsive correction is made. In this connection $u < 0$, because $x \geqslant 0$ in the entire region D, see (2.14). It follows from equations (1.1) and (1.5) that for impulsive correction (u is a delta function of time) in the case $m = 1$ and $u < 0$ the quantity $x - a(t)q$ does not change at the instant of correction. Consequently, the argument $x - a_1(\tau)p$ of function G in (3.4) also will not change. That is, in the region D_2 the phase point (τ, x, p) moves along the characteristic of equation $Q = 0$ in the process of correction. This is represented qualitatively in Fig. 1, where the section of region D by the plane $\tau =$ const is shown. The parallel straight lines in Fig. 1 are characteristics $x - a_1(\tau)p =$ const, on which the function S is constant in region D_2. If the phase point (τ, x, p) is in the region D_1, then $u = 0$. If, however, (τ, x, p) is located in D_2, then the control represents an impulse which instantaneously displaces the phase point along the straight lines (Fig. 1) in the direction indicated by the arrows (in the direction of decreasing x and p) to the boundary Γ or to the boundary of the region D. It is apparent that for the design of the optimal control system for $m = 1$ it is sufficient to determine the boundary Γ.

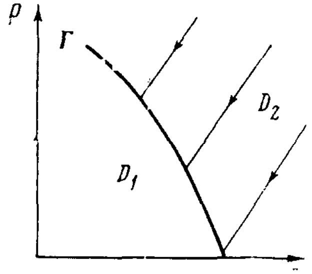

Fig. 1

We note one simple case which allows a simple exact solution. Let $a(T) \geqslant a(t)$ for $t \leqslant T$, i.e. the effectiveness of the control at the end of the process is maximal. Then the optimal control apparently represents an impulse delivered at the instant T. This impulse decreases the final deviation $|x(T)|$ to the extent allowed by the control capability.

Using the notation of Sect. 1, we have

$$u(t) = [x(T) - x(T-0)]\, a^{-1}(T)\, \delta(t - T) \tag{3.7}$$
$$x(T) = \max\,[|x(T-0)| - a(T)\, q_0,\ 0]\ \operatorname{sign} x(T-0)$$

In the notation of (2.7) it follows from Eqs. (3.7) that for $\tau \to +0$ we obtain

$$S(\tau, x, p) = F\,\{\max\,[x - a_1(0)\,p,\ 0]\},\quad x \geqslant 0,\ \tau \to +0 \tag{3.8}$$

Here the properties (1.4) have been utilized. The control relationship (3.7) shows that here the region D_2 degenerates into the plane $\tau = 0$ on which S experiences a jump from the value $F(x)$ to the value (3.8). The entire region D for $\tau > 0$ will be region D_1. In region D_1 we find the function S by writing the solution of (3.3) with the initial condition (3.8) and the symmetry condition (2.9)

$$S(\tau, x, p) = \frac{1}{\sqrt{2\pi\tau}} \int_{a_1(0)p}^{\infty} \left\{\exp\left[-\frac{(x - x_1)^2}{2\tau}\right] + \exp\left[-\frac{(x + x_1)^2}{2\tau}\right]\right\} \times$$
$$\times\, F(x_1 - a_1(0)\,p)\, dx_1,\ p \geqslant 0,\ \tau > 0 \tag{3.9}$$

In order to check the inequality $Q < 0$, we substitute solution (3.9) into Eq. (3.1)

$$Q = \frac{1}{\sqrt{2\pi\tau}} \int_{a_1(0)p}^{\infty} \left\{[a_1(\tau) - a_1(0)] \exp\left[-\frac{(x - x_1)^2}{2\tau}\right] - [a_1(0) + a_1(\tau)] \times\right.$$
$$\left.\times \exp\left[-\frac{(x + x_1)^2}{2\tau}\right]\right\} F'[x_1 - a_1(0)\,p]\, dx_1$$

From conditions (1.4) and $a_1(0) \geqslant a_1(\tau) > 0$ it follows that $Q < 0$ in the region D_1, so that Eq. (3.9) in fact gives a solution of the problem.

4. Self-similar solutions. Let the following equations be satisfied for $m \geqslant 1$:

$$F(x) = |x|^n,\quad a_1(\tau)\,[b_1(\tau)]^{2(1-m)/m} = A\tau^k,\quad n > 0,\quad A > 0 \tag{4.1}$$

where n, A, k are constants. In particular, if functions $a(t)$ and $b(t)$ in Eq. (1.1) have the form

$$a(t) = A_1 (T - t)^{\alpha},\qquad b(t) = B_1 (T - t)^{\beta}$$

where A_1, B_1, α, β are constants, then after substitution (2.7) we arrive at the condition (4.1) for

$$k = [\alpha + 2(1 - m)m^{-1}\beta]\,(2\beta + 1)^{-1}$$

If conditions (4.1) are satisfied, then Eq. (2.8) and boundary conditions (2.9), (2.10) for the case $m > 1$, and also relationships (3.3) and (3.4) for $m = 1$ will be invariant with respect to the following single-parameter group of expansion transformations:

$$x \to Cx,\quad \tau \to C^2\tau,\quad p \to C^{-r}p,\quad S \to C^n S,\quad r = 2k + 1 - 2m^{-1} \tag{4.2}$$

with the parameter C. Consequently, the boundary value problems in Sects. 2 and 3 have self-similar solutions which are invariant with respect to the group of transformations (4.2). These solutions can be sought for example in the following form:

$$S = \tau^{1/_2 n}\psi(y, z), \quad y = Ap\tau^{1/_2 r}, \quad z = x\tau^{-1/_2}, \quad r = 2k + 1 - 2m^{-1} \tag{4.3}$$

or in other equivalent forms. Substituting relationships (4.1), (4.3) into Eq. (2.8) and equalities (2.9), (2.10), (2.12), (2.13), we obtain an equation, boundary conditions and asymptotic representation for function $\psi(y, z)$ in the form

$$n\psi - z\psi_z + ry\psi_y = \psi_{zz} - 2\,[(m-1)/m]y\,[\psi_z^m/(-\psi_y)]^{1/(m-1)}$$
$$\psi_z(y, 0) = 0, \qquad \psi(0, z) = h_n(z) \tag{4.4}$$
$$\psi(y, z) = z^n\left[1 - \frac{ny}{z}\left(\frac{m-1}{km+m-1}\right)^{(m-1)/m}\right] \quad (z \to \infty,\ yz^r = \text{const})$$

Here the following notation is introduced:

$$h_n(z) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\exp\frac{-(z-t)^2}{2}\,|t|^n\,dt =$$
$$= \frac{\Gamma(n+1)}{\sqrt{2\pi}}\exp\frac{-z^2}{4}\,[D_{-n-1}(z) + D_{-n-1}(-z)] \tag{4.5}$$

where D_{-n-1} are parabolic cylindrical functions [2] by which the integral (4.5) is expressed. For natural n Eq. (4.5) can be simplified [2] and assumes the form

$$h_n(z) = \exp\frac{-z^2}{2}\,\frac{d^n}{dz^n}\left\{\exp\frac{z^2}{2}\left[1 + \frac{(-1)^n - 1}{\sqrt{\pi}}\operatorname{Erfc}\left(\frac{z}{\sqrt{2}}\right)\right]\right\}$$
$$\operatorname{Erfc} z = \int_z^{\infty} e^{-t^2}\,dt = \frac{\sqrt{\pi}}{2} - \int_0^z e^{-t^2}\,dt \qquad (n = 1, 2, \ldots) \tag{4.6}$$

The boundary value problem (4.4) is reduced to the determination of function $\psi(y, z)$ in the region $y \geqslant 0$ and $z \geqslant 0$. This problem is much simpler than the initial problem (2.8)–(2.10), because it contains only two independent variables.

Analogous simplifications are carried out for the case $m = 1$. Substituting Eqs. (4.1) and (4.3) for $m = 1$ into relationships (3.3), (3.4), (3.6), (2.9), (2.10) we have the following boundary value problem:

$$n\psi - z\psi_z + (2k-1)\,y\psi_y = \psi_{zz}, \quad Q^\circ < 0 \quad \text{in } D_1^\circ$$
$$\psi = G^\circ(z - y), \quad Q^\circ = 0 \quad \text{in } D_2^\circ \tag{4.7}$$
$$Q^\circ = Q_z^\circ = 0 \quad \text{on } \Gamma^\circ, \qquad \psi_z(y, 0) = 0, \quad \psi(0, z) = h_n(z)$$
$$\psi(y, z) = z^n \quad \text{for} \quad z \to \infty, \quad yz^{2k-1} = \text{const}, \qquad Q^\circ = \psi_z + \psi_y$$

Here D_1° and D_2° are regions of the quadrant $y \geqslant 0,\ z \geqslant 0$ in which the conditions $Q^\circ < 0$ and $Q^\circ = 0$ are satisfied, respectively. Here Γ° is the unknown boundary which separates these regions. An arbitrary function of one variable is denoted by G°. This function is subject to determination. From the inequality $S_p < 0$ it follows that $\psi_y < 0$ everywhere. Therefore on the straight line $z = 0$ where $\psi_z = 0$ we have $Q^\circ < 0$ in accordance with (4.8). Consequently, the region D_1° contains the straight line $z = 0$.

The boundary value problem with an unknown boundary (4.7) allows an exact analytical solution for $k = 0$ and $k = 1/_2$. If $k = 0$ and $m = 1$ in the relationship (4.1), the condition $a_1(\tau) = A$ is satisfied. In this connection it follows from Sect. 3 that solution (3.9) is appropriate. Passing in (3.9) to variables (4.3) for $k = 0$ and $m = 1$, we obtain the following exact solution of problem (4.7) for $k = 0$:

$$\psi(y, z) = \frac{1}{\sqrt{2\pi}} \int_y^\infty \left[\exp\frac{-(z-t)^2}{2} + \exp\frac{-(z+t)^2}{2}\right] |y-t|^n \, dt \qquad (4.8)$$

The region D_1° in which the solution (4.8) is applicable and the inequality $Q^\circ < 0$ is satisfied, coincides here with the entire quadrant $y \geqslant 0$, $z \geqslant 0$. The region D_2° degenerates going to infinity.

Let us examine the case $k = 1/2$. For $k = 1/2$ we obtain for the region D_1° from (4.7)

$$n\psi - z\psi_z = \psi_{zz}, \qquad \psi_z(y, 0) = 0 \qquad (4.9)$$

By a direct verification it is easy to become convinced that the function $h_n(z)$ from (4.5) satisfies Eqs. (4.9). Consequently, the solution in the region D_1° has the form

$$\psi(y, z) = h_n(z) g_n(y) \qquad (4.10)$$

where g_n so far is an arbitrary function. Substitution (4.10) into the boundary condition (4.7) on the boundary Γ° yields

$$h_n' q_n + h_n g_n' = 0, \qquad h_n'' g_n + h_n' g_n' = 0 \qquad (4.11)$$

Primes indicate derivatives with respect to arguments y and z. For the existence of a nontrivial solution $g_n(y)$ the determinant of system (4.11), which is linear and homogeneous with respect to g_n and g_n', must be equal to zero. Hence, we obtain

$$h_n''(z) h_n(z) = h_n'^2(z) \qquad (4.12)$$

We denote the smallest positive root of Eq. (4.12) by z_n. In this manner the boundary Γ° for $k = 1/2$ is the straight line $z = z_n$. From the first equation (4.11) we find

$$g_n(y) = \exp[-h_n'(z_n) y / h_n(z_n)] \qquad (4.13)$$

Here the condition $g_n(0) = 1$ was used. This condition results from the boundary condition (4.7) for $y = 0$ and from the relationship (4.19). Hence, in the region D_1° which is determined by the condition $0 \leqslant z < z_n$, the solution $\psi(y, z)$ is given by Eqs. (4.10), (4.13) in which the function $h_n(z)$ is determined by Eqs. (4.5), (4.6) and the value z_n is determined by Eq. (4.12). In the region D_2°, given by the inequality $z > z_n$, we have $\psi = G^\circ(z - y)$ in accordance with (4.7).

Let us set $z = z_n$ and equate the solution in regions D_1° and D_2°. We obtain

$$h_n(z_n) g_n(y) = G^\circ(z_n - y), \qquad y \geqslant 0$$

Using (4.13), we find from the above

$$G^\circ(x) = h_n(z_n) \exp[h_n'(z_n)(x - z_n) / h_n(z_n)], \qquad x \leqslant z_n \qquad (4.14)$$

For the determination of function $G^\circ(x)$ when $x > z_n$ we take advantage of the boundary condition (4.7) for $y = 0$. Denoting the argument of function G° by x, we obtain

$$G^\circ(x) = h_n(x), \qquad x \geqslant z_n \qquad (4.15)$$

Equations (4.14) and (4.15) completely describe the function G° and by the same token the solution in the region D_2°. In this manner, the solution of the problem for $k = 1/2$ is fully constructed, and it is possible to show that it satisfies all condition (4.7). By equations (4.3) we can return to the initial variables.

If the variables τ, x, p are such that $z < z_n$, the control must be equal to zero (region D_1°, see Sect. 3). If however $z > z_n$, the impulsive control should be carried out. In this connection for $z_n < z < z_n + y$ the impulse must transfer the system to the state $z = z_n$ and for $z_n \geqslant z_n + y$ to the state $y = 0$. In the latter case the entire control capability is expended at once.

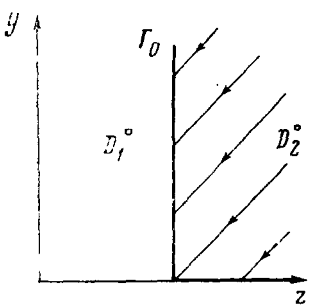

Fig. 2

In Fig. 2 the regions D_1°, D_2° and the straight lines $y - z = \text{const}$ are represented. On these straight lines the function ψ is constant in the region D_2^0 and the phase point moves along these lines during impulsive control.

As an example let us examine the particular case $k = 1/2$, $n = 2$, which corresponds to a quadratic function $F(x)$ in the criterion (1.3). From relationships (4.6), (4.12), (4.13) we find for $n = 2$

$$h_2(z) = z^2 + 1, \qquad z_2 = 1, \qquad g_2(y) = e^{-y} \tag{4.16}$$

The solution of (4.10), (4.14), (4.15) for $n = 2$, taking into account (4.16), assumes a simple form

$$\psi(y, z) = (z^2 + 1)e^{-y} \quad (z \leqslant 1), \qquad \psi(y, z) = 2e^{z-y-1} \quad (1 \leqslant z \leqslant 1 + y)$$

$$\psi(y, z) = (z - y)^2 + 1 \qquad (z \geqslant 1 + y)$$

In the general case ($k \neq 0$, $k \neq 1/2$, $m \neq 1$) the problems considered here, can be solved numerically, for example by the method of finite differences; some computations of this type are carried out at the Institute of Problems in Mechanics, Academy of Sciences, USSR.

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